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On Weakly Pure Submodules of Multiplication Modules

Arvind Kumar Sinha[1], Biswajit Barik[2]

Department of Mathematics, National Institute of Technology Raipur, INDIA[1, 2]

Email: aksinha.maths@nitrr.ac.in[1] , *biswajit.nitr@gmail.com*[2]

Abstract- In this paper we investigate some properties of weakly pure submodules of multiplication modules.

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1. INTRODUCTION

Multiplication module was introduced by Barnard [4] in 1981 and after a long time pure submodules of multiplication modules has been investigated by Ali and Smith [1] in 2004. The concept of pure submodule of multiplication module Khaksari [5] has investigated weakly pure submodules of multiplication modules in 2011.

A submodule N of R- module M is called pure if $IN = N \cap IM$, for every ideal I of R. An R- submodule N of R- module M is called weakly pure if $IN = N \cap IM$, for every Boolean ideal I of R.

If R is a ring and N is a submodule of an R- module M, the ideal $\{r \in R: rM \subseteq N\}$ will be denoted by $(N:M)$. An R- module M is called multiplication module if for every submodule N of M there exist an ideal I of R such that $N = IM$.

The aim of this paper is to introduce some results on weakly pure submodules of multiplication modules by using results in Ali and Smith [1] and Atani and Ghaleh [2].

2. PRELIMINARIES

Throughout this paper all rings will be commutative with non-zero identity and all modules will be unitary.

In this section we give some basic definitions and proofs of some lemma related to weakly pure submodules of multiplication modules. We begin with the prove of following well-known results, it will help to prove our main results.

Definition 2.1. [5] Let $\boldsymbol{M}$ be a module over a ring $\boldsymbol{R}$. A proper submodule $\boldsymbol{N}$ of $\boldsymbol{M}$ is said to be prime if $\boldsymbol{rm} \in \boldsymbol{N}$ for $\boldsymbol{r} \in \boldsymbol{R}$ and $\boldsymbol{m} \in \boldsymbol{M}$ implies that either $\boldsymbol{m} \in \boldsymbol{N}$ or $\boldsymbol{r} \in (\boldsymbol{N}:\boldsymbol{M})$.

Definition 2.2 [5] An ideal I or R is called Boolean ideal if every element of I is idempotent.

Definition 2.3 [5] R- submodule N of R- module M is called weakly pure if $IN = N \cap IM$, for every Boolean ideal I of R.

Lemma 2.1 [5] Let R be a commutative ring, M- a multiplication R- module and N a prime R- submodule of M. Then N is weakly pure.

Proof. Assume that I is a Boolean ideal of R. As $IN \subseteq N \cap IM$ is trivial, we shall prove the reverse inclusion, let $x \in N \cap IM$. Then there exist $r \in I$ and $m \in M$ such that $x = rm$. But $x \in N$ and N is prime, so $m \in N$ (it follows that $rm \in IN$) or $r \in (N:M)$. Hence $rm = r^2m \in I(N:M)M = IN$.

Lemma 2.2 [5] Let R be a commutative ring, M a free multiplication R- module and N a weakly pure R- submodule of M. Then $I(N:M) = I \cap (N:M)$ for every Boolean ideal I of R.

Proof. Assume that I is a Boolean ideal of R and let $\{x_i : i \in j\}$ be basis of M. Also $IN = N \cap IM$. As $I(N:M) \subseteq I \cap (N:M)$ is trivial, we shall prove the reverse inclusion. Let $a \in (N:M) \cap I$. Then $ax_i \in I(N:M)M$ because N is a weakly pure. It follows that there exist $r \in I, b \in (N:M)$ such that $ax_i = rbx_i$, hence $a = rb \in I(N:M)$ as required.

3. Main results

In this section, we introduce some properties of weakly pure submodules of multiplication modules.

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Proposition 3.1. Let R be a commutative ring, M a free multiplication R- module and N weakly pure R-submodule of M. Then the ideal $(N:M)$ is idempotent.

Proof. By lemma 2.2, we have $(N:M)^2 = (N:M) \cap (N:M) = (N:M)$.

Theorem 3.1 Let M be a free multiplication module over a commutative ring R. Then every weakly pure submodule of M is idempotent.

Proof. Let N be a weakly pure submodule of M. Then by proposition 3.1, we have $N^2 = (N:M)^2M = (N:M)M = N$.

Theorem 3.2 Let R be a commutative ring, M a free multiplication R- module and N weakly pure R-submodule of M. Then N is primary submodule of M if and only if it is weakly primary submodule of M.

Proof. It is enough to show that if N is weakly primary then N is primary. Assume that $N \neq 0$ is a weakly primary submodule of M that is not primary. Then by [3, proposition 2.14] and proposition 3.1, we have $N = (N:M)M = (N:M)^2M = (N:M)N = 0$, which is contradiction. Thus N is primary.

Proposition 3.2 Let R be a commutative Noetherian ring with Jacobson radical $J(R)$, M a free multiplication R- module and N weakly pure R-submodule of M. Then there is a maximal ideal P of R such that $(N:M) \nsubseteq P$.

Proof. Otherwise, $(N:M) \subseteq J(R)$, so $(N:M) = \bigcap_{i=1}^{\infty}(N:M)^i = 0$ by proposition 3.1, hence $N = (N:M)M = 0$, which is contradiction. Hence there is a maximal ideal P of R such that $(N:M) \nsubseteq P$.

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